Almost-complex structures

Def: A <u>complex structure</u> on a real vect. space V is $J: V \to V$, $J^2 = -Id$.
($\Rightarrow$ cx. vect space using $iv := Jv$)

J is <u>compatible</u> with a sympl. form Ω if $G(u,v) := \Omega(u, Jv)$ is a definite positive symmetric inner product.

ie. • $\Omega(Ju, Jv) = \Omega(u,v)$ ($\Leftrightarrow G$ symmetric)
ie. $J \in Aut(V, \Omega)$.

• $\Omega(u, Ju) > 0 \quad \forall u \neq 0$ (say J <u>tames</u> Ω).

Fact: $(V, \Omega, J) \simeq (\mathbb{R}^{2n}, \omega_0, J_0)$ + can arrange: a given Lag. subspace $L \subset V \mapsto \mathbb{R}^n_x \subset \mathbb{R}^{2n}$

<u>Pf:</u> construct a standard basis:
pick $e_1 \in V$, scale so $\Omega(e_1, Je_1) = 1$, set $f_1 = Je_1$. (rel. case: take $e_1 \in L$)
Then J compat. $\Rightarrow$ produces sympl. orthogonal W to span(e_1, f_1). (and $L \cap W$ Lagrangian in W)
Induction on dim. $\Rightarrow$ find std basis $(e_2, f_2 = Je_2, \dots, e_n, f_n = Je_n)$ of W. (with $L \cap W = span(e_j)$)

Prop: (V, Ω) sympl. vector space $\Rightarrow \exists$ compatible J
In fact given any positive inner product $\langle \cdot, \cdot \rangle$, can build canonical Ω-compatible J.
(but $G = \Omega(\cdot, J\cdot)$ isn't $\langle \cdot, \cdot \rangle$ unless it was compat.)

<u>Pf:</u> $u \mapsto \Omega(u, \cdot)$ and $\langle u, \cdot \rangle$ are both isos. $V \xrightarrow{\sim} V^*$, so $\exists A: V \to V$ st. $\Omega(u,v) = \langle Au, v \rangle$.
(in $\langle \cdot, \cdot \rangle$-orthonormal basis, A = matrix of Ω)
A is invertible, and skew symmetric ($\Omega(v,u) = \langle Av, u \rangle = \langle v, A^* u \rangle = \langle A^* u, v \rangle = -\langle Au, v \rangle$)
hence $AA^* = -A^2$ is symmetric definite positive ($\langle AA^* u, u \rangle = \langle A^* u, A^* u \rangle > 0 \ \forall u \neq 0$)
AA^* diagonalizable with >0 eigenvalues $\Rightarrow \exists \sqrt{AA^*} = diag(\sqrt{\lambda_i})$ symm. def. positive.
Let $J = (\sqrt{AA^*})^{-1} A$. (the decomp^n $A = \underset{\text{symm.} > 0}{\sqrt{AA^*}} \ \underset{\text{orthogonal}}{J}$ is the <u>polar decomp.</u> of A).

Now, A commutes with $\sqrt{AA^*}$:
Indeed: $v \in V_i$ eigenspace of $AA^* = -A^2$ for λ_i (resp. $\sqrt{\ }$ for $\sqrt{\lambda_i}$).
$\Rightarrow (AA^*)Av = -A^3 v = A(AA^*)v = \lambda_i Av \Rightarrow A$ preserves $V_i \Rightarrow$ commutes.

Hence J also commutes with A. We get:

$$\left.\begin{array}{l} \bullet\ J^* = A^* (\sqrt{AA^*})^{-1} = -(\sqrt{AA^*})^{-1} A = -J \quad \text{skewsymmetric} \\ \bullet\ J^* J = A^* (\sqrt{AA^*})^{-1} (\sqrt{AA^*})^{-1} A = Id \quad \text{ie. } J \text{ is orthogonal} \end{array}\right\} \Rightarrow J^2 = -J^* J = -1 \ \checkmark \ \text{cx. structure}$$

Compatibility: $\Omega(u, Jv) = \langle Au, Jv \rangle = \langle -JAu, v \rangle = \langle -(\sqrt{AA^*})^{-1} A^2 u, v \rangle$
$= \langle \sqrt{AA^*}\, u, v \rangle$ ↳ symm. definite positive ✓

(and the metric $G(u,v)$ is $\langle \sqrt{AA^*}\, u, v \rangle$) $\square$

<u>Rmk:</u> if $\langle \cdot, \cdot \rangle$ was Ω-compatible, in the sense that A was a cx structure, then $\sqrt{AA^*} = 1$, $G = \langle \cdot, \cdot \rangle$, $J = A$.

Def: an almost-complex structure on M is $J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{id}$.
It is compatible (resp. tame) with ω if $\forall x \in M$, J_x is ω_x-compatible (tame)
Then $g = \omega(\cdot, J\cdot)$ Riemannian metric.

Corollary: Every sympl. mfld (M,ω) admits compatible a.c.s. The space $\mathcal{J}(M,\omega)$ of these is path connected (in fact contractible).

Pf.
- pick some g_0 Riem. metric $\Rightarrow$ using polar decomposition, get J compatible a.c.s.
- given J_0, J_1 ω-compatible acs $\rightsquigarrow$ g_0, g_1 the corresponding Riem. metrics,
  $g_t := t g_1 + (1-t) g_0$ Riem metrics $\forall t \in [0,1]$, polar decomp^n $\Rightarrow$ $\tilde{J}_t$ compatible acs
  (with $\tilde{J}_0 = J_0$, $\tilde{J}_1 = J_1$).

(contraction = use this to homotope id of $\mathcal{J}(M,\omega)$ to constant map J_0).

Rmk: we didn't use $d\omega = 0$, these are all pointwise arguments & apply on any sympl. v.b., not just tangent bundle of a sympl. mfld. Hence classif^n of sympl. v.b $\Leftrightarrow$ classif^n of cx. vect. bundles.

Rmk: (M,J) almost-cx manifold $\Rightarrow$ the space of J-compatible symplectic forms is convex if non-empty

ie: ω_0, ω_1 symplectic J-compat. $\Rightarrow$ $t\omega_1 + (1-t)\omega_0$ is symplectic & J-compat. $\forall t \in [0,1]$.

(but may be empty! e.g. S^6 has an a.c.s., $S^1 \times S^3$ has a cx. structure, these are not symplectic manifolds! linear algebra gives a nondeg compatible 2-form but never a closed one. In fact, $\exists$ a.c.s. $\Leftrightarrow$ $\exists$ nondegenerate 2-form, by the above.)

★ **Compatible triples:** any two of (J, g, ω) determine the third one!
So preserving two of these structures $\Leftrightarrow$ preserving all three. On $\mathbb{R}^{2n}$: in $GL(2n,\mathbb{R})$,

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n,\mathbb{C}) = O(2n) \cap GL(n,\mathbb{C}) = U(n).$$

★ Since every ω_0-compat. cx structure on $\mathbb{R}^{2n}$ is standard, ie. $\exists\, \phi : (\mathbb{R}^{2n}, \omega_0, J) \xrightarrow{\sim} (\mathbb{R}^{2n}, \omega_0, J_0)$
$\phi \in Sp(2n)$ ie. $\phi^* \omega_0 = \omega_0$

we get: space of compat. cx. str. $\mathcal{J}(2n) \cong Sp(2n)/U(n)$
(has transitive $Sp(2n)$-action with stabilizer of $J_0 = U(n)$)

Know this is contractible. Hence: $Sp(2n)$ retracts onto max. compact subgroup $U(n)$

★ Similarly, $U(n)$ acts transitively on $LGr(n) = \{\text{Lagr. } n\text{-planes in } (\mathbb{R}^{2n}, \omega_0)\}$,
stabilizer of $L_0 = \mathbb{R}^n_x$ is $O(n)$, hence: $\underline{LGr(n) \cong U(n)/O(n)}$.

Type decomposition:

- $J^2=-1$ extend linearly to $TM\otimes\mathbb{C} \Rightarrow$ splits into eigenspaces for $+i$: $TM^{1,0}$
  $-i$: $TM^{0,1}$

$$\left.\begin{aligned} \pi^{1,0} : TM &\longrightarrow TM^{1,0} \\ v &\longmapsto \tfrac{1}{2}(v-iJv) = v^{1,0} \\ \pi^{0,1} : TM &\longrightarrow TM^{0,1} \\ v &\longmapsto \tfrac{1}{2}(v+iJv) = v^{0,1} \end{aligned}\right\}$$

$\mathbb{R}$-bundle isomorphisms, $v = v^{1,0}+v^{0,1}$.
$(Jv)^{1,0} = i v^{1,0}$
$(Jv)^{0,1} = -i v^{0,1}$ ie. $(TM,J) \simeq TM^{1,0} \simeq \overline{TM^{0,1}}$ as complex vect. bundles.

- on cotangent bundle: $T^*M^{1,0} = \{\eta \in T^*M\otimes\mathbb{C} / \eta(Jv) = i\eta(v)\}$ $\mathbb{C}$-linear 1-forms
  $T^*M^{0,1} = \{$ ——— " ——— $-i\}$ $\mathbb{C}$-antilinear

$$\eta^{1,0} = \tfrac{1}{2}(\eta - i(\eta\circ J)) = \tfrac{1}{2}(\eta + i J^*\eta)$$

where $J^*: T^*M \longrightarrow T^*M$, $\eta \longmapsto -\eta\circ J$ "adjoint to J for compat. metric

- On differential forms: $\Lambda^k T^*M\otimes\mathbb{C} \cong \bigoplus_{p+q=k} \Lambda^{p,q}T^*M$

$\Lambda^{p,q} = (\Lambda^p T^*M^{1,0})\otimes(\Lambda^q T^*M^{0,1})$ pairs nontrivially with p $(1,0)$ vector fields & q $(0,1)$ ——"——
trivially with all other combinations.

Ex: on a complex manifold ie. $\exists$ charts $U_\alpha \xrightarrow[\varphi_\alpha]{} \mathbb{C}^n$ st. $(\varphi_\alpha)_* J = J_0$, in local coords.

$z_j = x_j + iy_j$: $TM^{1,0} = \mathrm{span}\left(\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\right)$, $TM^{0,1} = \mathrm{span}\left(\frac{\partial}{\partial \bar z_j}\right)$

$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$

$J^* dx_j = dy_j$

$T^*M^{1,0} = \mathrm{span}(dz_j = dx_j + i\,dy_j)$, $T^*M^{0,1} = \mathrm{span}(d\bar z_j)$.

- Given a function $f: M\to\mathbb{C}$, type decomp. $df = (df)^{1,0} + (df)^{0,1} =: \partial f + \bar\partial f$.

On a complex manifold, $d = \partial + \bar\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}\oplus\Omega^{p,q+1}$
but on an almost cx. manifold d doesn't behave as well w.r.t type decomposition!

Def: an almost-complex structure is integrable if $[TM^{1,0}, TM^{1,0}] \subseteq TM^{1,0}$
↑ cx. linear extension of Lie bracket.

Nijenhuis tensor:

$$N(u,v) = [Ju,Jv] - J[u,Jv] - J[Ju,v] - [u,v].$$
$$= -8\,\mathrm{Re}\left([u^{1,0},v^{1,0}]^{0,1}\right).$$

indeed, $[u^{1,0},v^{1,0}] = \frac{1}{4}[u-iJu, v-iJv] = \frac{1}{4}\left([u,v] - [Ju,Jv] - i[Ju,v] - i[u,Jv]\right)$
taking 0,1 part then real part turns the $-i$'s into $+J$'s

So: $N=0 \iff J$ is integrable. ie. $[\,,\,]$ preserves splitting of $TM\otimes\mathbb{C}$.

Prop: N is a tensor (doesn't involve derivatives of u,v)
skew symmetric
$\mathbb{C}$-antilinear ($N(Ju,v) = -JN(u,v)$).

ie. can view as a linear map $N: \Lambda^2 T^{1,0} \to T^{0,1}$, or $N \in \Omega^{2,0} \otimes T^{0,1}$.

Consequence: if $\dim_{\mathbb{R}} M = 2$ then $N \equiv 0$: every a.c.s. on a surface is integrable.

Thm (Newlander-Nirenberg): J is a complex structure (ie. $\exists$ loc. holomorphic coordinates) iff $N \equiv 0$.

This is clearly a necessary condition: in local cx. coords, should have $\left[\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right] = 0$.
Otherwise, $\nexists$ local holom. coords!

In fact: $\alpha \in \Omega^{0,1} \Rightarrow d\alpha = (d\alpha)^{2,0} + \underset{1,1}{\partial\alpha} + \underset{0,2}{\bar\partial\alpha}$, what is $(d\alpha)^{2,0}$?

Answer: $d\alpha^{(2,0)}(u,v) = d\alpha(u^{1,0}, v^{1,0}) = L_{u^{1,0}}\alpha(v^{1,0}) - L_{v^{1,0}}\alpha(u^{1,0}) - \alpha([u^{1,0}, v^{1,0}])$

0 since $\alpha \in \Omega^{0,1}$

$= -\alpha([u^{1,0}, v^{1,0}]^{0,1}) = N^*\alpha$

↳ tensor dual to N. $\Lambda^{0,1}T^*M \to \Lambda^{2,0}T^*M$.

In particular: $f: M \to \mathbb{C}$
look at $(0,2)$-part of $d^2 f = 0 \Rightarrow \bar\partial(\bar\partial f) + d(\partial f)^{0,2} = 0$
ie. $\bar\partial(\bar\partial f) = -\bar N^*(\partial f)$.

Thus: f J-holomorphic $\Rightarrow \bar N^*(\partial f) = 0$.

If N is nondegenerate then $\nexists$ nonconstant holom. functions!
(eg. for generic J in $\dim_{\mathbb{C}} > 2$).

Def: Kähler manifold $= (M, \omega, J)$, ω symplectic, J compatible & integrable.

(Then on Kähler manifolds, $d = \partial + \bar\partial$, Hodge theory $\rightsquigarrow$ using rel$^{\underline{n}}$ between Riemannian Laplacian and $\bar\partial$-Laplacian, get that harmonic forms split by types and hence $H^k(M,\mathbb{C}) = \bigoplus_{p+q} H^{p,q}(M)$ ... we're close to alg. geometry, in fact a compact Kähler mfld with $[\omega] \in H^2(M,\mathbb{Z})$ integer is a cx. proj. variety.)

Kähler potentials / spsh functions:

(M,J) complex or almost-complex.

Def: $f: M \to \mathbb{R}$ is strictly plurisubharmonic (spsh) if $dd^c f(v, Jv) > 0 \ \forall v \neq 0$.

here $d^c f := -df \circ J = J^* df$

Ex: in the complex plane, $f = \frac{1}{4}|z|^2 = \frac{1}{4} r^2 \Rightarrow$
$$df = \tfrac{1}{2} r\, dr$$
$$d^c f = \tfrac{1}{2} r^2 d\theta$$
$$dd^c f = r\, dr \wedge d\theta = dx \wedge dy.$$

On a complex manifold: $(d = \partial + \bar\partial,\ \partial\bar\partial + \bar\partial\partial = 0,\ \partial^2 = \bar\partial^2 = 0)$
$$d^c f = J^*(\partial f + \bar\partial f) = i(\partial - \bar\partial) f, \text{ so } dd^c f = -2i\, \partial\bar\partial f, \text{ is of type } (1,1).$$

This is relevant because:

1) $dd^c f$ is an exact symplectic form, taming J; compatible w/ J if J integrable.
   closed since exact; non-deg since $\omega(v, Jv) > 0$.

   Conversely, every Kähler form can locally be expressed by a spsh Kähler potential $\omega = dd^c f$.
   on a subset where ω is exact

2) $u: (\Sigma^2, j) \to (M, J)$ J-holomorphic (ie. $J \circ du = du \circ j$) $\Rightarrow$

   $d^c(f \circ u) = j^*(u^* df) = u^*(J^* df) = u^*(d^c f)$ and similarly $dd^c(f \circ u) = u^*(dd^c f)$
   is a positive area form. But... in complex dim. 1, $dd^c f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$
   $z = x + iy$

   Along complex curves, $dd^c f$ = Laplacian

   And so $dd^c f > 0 \Rightarrow f \circ u$ satisfies a maximum principle along J-hol. curves
   namely $f \circ u$ cannot have a local max.

   This is used to control behavior of J-holom. curves in noncompact symp. manifolds.

* **Note:** toric Kähler manifolds admit local Kähler potentials that are invariant under T^n (by averaging) $\Rightarrow f$ = function of the norms of coordinates. (or of the moment map).

Open stratum of a complex toric variety is $\simeq (\mathbb{C}^*)^n$, let $z_j = e^{\rho_j + i\theta_j}$ $\circlearrowleft T^n$ acts by rotation

Then $f = f(\rho_1, \dots \rho_n) \Rightarrow dd^c f = \sum \frac{\partial^2 f}{\partial \rho_i \partial \rho_j} d\rho_i \wedge d\theta_j = \sum d\left(\frac{\partial f}{\partial \rho_j}\right) \wedge d\theta_j$

is spsh iff $\left(\frac{\partial^2 f}{\partial \rho_i \partial \rho_j}\right)$ definite positive ie. f convex as function of the $\rho = \log|z|$
and the moment map is $(\mu_1, \dots \mu_n) = \left(\frac{\partial f}{\partial \rho_1}, \dots, \frac{\partial f}{\partial \rho_n}\right)$.

<u>Ex:</u> · Kähler potential on $(\mathbb{C}^n, \omega_0)$: $\omega_0 = dd^c\left(\frac{1}{4}\|z\|^2\right)$.

· on $(\mathbb{CP}^n, \omega_{std})$: $\omega_{std} = dd^c\left(\frac{1}{4}\log\left(|z_0|^2 + \dots + |z_n|^2\right)\right)$ (even though not globally exact !!).

↳ calculate on $\mathbb{C}^{n+1} \setminus \{0\}$, find it's scale inv't and lifted from $\mathbb{CP}^n$ under projection.

or ... in affine charts $\{(1 : z_1 : \dots : z_n)\} \subset \mathbb{CP}^n$, $dd^c\left(\frac{1}{4}\log\left(1 + |z_1|^2 + \dots + |z_n|^2\right)\right)$

(change of charts to eg. $\left(\frac{1}{z_j} : \frac{z_1}{z_j} : \dots\right)$ the two local potentials differ by $\frac{1}{4}\log(|z_j|^2) = \frac{1}{2}\rho_j$ which is harmonic).

logarithmic derivatives recover known formula for moment map.

· <u>Def:</u> $u : (\Sigma, j) \longrightarrow (M, J)$ is <u>J-holomorphic</u> if $du \circ j = J \circ du$, or

$$\bar{\partial}_J u := \frac{1}{2}\left(du + J \circ du \circ j\right) = 0 \in \Lambda^{0,1}(\Sigma, u^*TM)$$

complex antilinear 1-forms on Σ w/ vals. in the complex vector bundle u^*TM.

The image of such a map is then an almost-complex submfld (possibly singular) of M.

· For generic J, when $\dim_{\mathbb{C}} u(\Sigma) > 1$ this is an overdetermined problem ⇒ expect no J-hol. submflds other than curves

(but integrability changes things: complex mflds do contain complex submflds!)

· The definition makes sense in almost-complex geometry, but the theory of J-hol. curves is only well-behaved in the symplectic case (thanks to Gromov compactness).

When J is compat. (or even just tame), J-hol. submanifolds are always <u>symplectic</u> i.e. the restr. of ω is a symplectic form.

In fact, $u^*\omega$ is a closed 2-form, non degenerate at points where u is an immersion (since $u^*\omega(v, jv) = \omega(du(v), J\,du(v)) > 0$).

For a non constant holom. curve w/ domain Σ = closed surface,

$$\int_\Sigma u^*\omega = \langle \underset{H^2(M)}{[\omega]}, \underset{H_2(M)}{[u(\Sigma)]} \rangle > 0 \qquad (\Rightarrow \omega \text{ can't be exact, and } \Sigma \text{ can't be null-homologous}).$$

Similarly for Lagrangian boundary, area = pair $[\omega] \in H^2(M, L; \mathbb{R})$ with $[u(\Sigma)] \in H_2(M, L)$.

This is also equal to the <u>area</u> of $u(\Sigma)$ for the compatible metric $g = \omega(\cdot, J\cdot)$.

(because: $\omega(v, Jv) = \|v\|^2 =$ g-area of Jv

).